

APPLICATIONS OF THE STRONG SPLITTER THEOREM: DECOMPOSITION RESULTS

S. R. Kingan ¹

Department of Mathematics

Brooklyn College, City University of New York

Brooklyn, NY 11210

skingan@brooklyn.cuny.edu

ABSTRACT. We use the Strong Splitter Theorem to decompose the excluded minor class of binary matroids with no E_4 -minor. Using this theorem we can get the 3-decomposers and the extremal internally 4-connected matroids as well as any other important matroids in the class. The matroid E_4 is a self-dual 10-element binary 3-connected matroid that plays a useful role in structural results. It is a single-element coextension of P_9 , which is a single-element extension of the 4-wheel. We show that the extremal matroids in this class are the binary rank- r spikes Z_r , the rank 3 and 4 projective geometries F_7 and $PG(3, 2)$, respectively, the 17-element internally 4-connected matroid R_{17} , and one 12-element rank-6 matroid. All the other 3-connected members have P_9 or P_9^* as 3-decomposers. As immediate corollaries we get decomposition results for $EX[P_9^*]$ and $EX[P_9]$ as well as the internally 4-connected members of these classes.

1. Introduction

The Splitter Theorem states that, if N is a 3-connected proper minor of a 3-connected matroid M such that, if N is a wheel or whirl then M has no larger wheel or whirl, respectively, then there is a sequence M_0, \dots, M_n of 3-connected matroids with $M_0 \cong N$, $M_n = M$, and for $i \in \{1, \dots, n\}$, M_i is a single-element extension or coextension of M_{i-1} [8]. The Strong Splitter Theorem imposes an ordering on the way N can be extended and coextended. We can obtain up to isomorphism M starting with N and at each step doing a 3-connected single-element extension or coextension, such that at most two consecutive single-element extensions occur in the sequence (unless the rank of the matroids involved is $r(M)$). Moreover, if two consecutive single-element extensions by elements $\{e_1, e_2\}$ are followed by a coextension by element f , then $\{e_1, e_2, f\}$ form a triad in the resulting matroid [3]. The Strong Splitter Theorem gives the most efficient way of generating a class of 3-connected matroids, thereby reducing computations.

Let $EX[M_1 \dots, M_k]$ denote the class of 3-connected binary matroids with no minors isomorphic to M_1, \dots, M_k . In this paper we demonstrate the usefulness of the Strong Splitter Theorem by characterizing an excluded minor class. The general strategy in our characterization is to find 3-decomposers and the extremal 3-connected matroids that cannot be decomposed. Note that it is sufficient to focus on the 3-connected members of an excluded minor class, since matroids that are not 3-connected can be pieced together from 3-connected matroids using the operations of 1-sum and 2-sum. Moreover, when we exclude a non-regular matroids all the regular matroids are automatically in the class. Thus, in the statement of our theorems we identify only the 3-connected non-regular matroids.

A matrix representation for P_9 is given below. It is a single-element extension of the 4-wheel. The matroid P_9^* is its dual and the matroid E_4 is a single-element extension of P_9^* .

¹The author is partially supported by PSC-CUNY grant number 66305-00 44.

$$P_9 = \left[\begin{array}{c|ccccc} I_4 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array} \right] P_9^* = \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 & 0 \end{array} \right] E_4 = \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

The main theorem in this paper characterizes $EX[E_4]$. As immediate corollaries we get characterizations for $EX[P_9^*]$ and $EX[P_9]$. Observe that F_7 and $PG(3, 2)$ shown below are in $EX[E_4]$ as well as in $EX[P_9^*]$ because they have rank 3 and 4, respectively, and E_4 and P_9^* have rank 5.

In [6] Oxley proved that a 3-connected binary non-regular matroid M has no minor isomorphic to P_9 or P_9^* if and only if M is isomorphic to F_7 , F_7^* , Z_r , Z_r^* , $Z_r \setminus b_r$, or $Z_r \setminus c_r$, for some $r \geq 4$. The matroid Z_r is represented by the matrix $[I_r | D]$ where D has $r + 1$ columns labeled $b_1 \dots b_r, c_r$ with zeros on the diagonal and ones elsewhere. The matroids $Z_r \setminus b_r$ and $Z_r \setminus c_r$ are its 3-connected single-element deletion-minors and $Z_r \setminus \{b_r, c_r\} \cong Z_{r-1}^*$. Moreover, $Z_r \setminus b_r$ and $Z_r \setminus c_r$ are self-dual. Thus, the infinite family Z_r for $r \geq 4$ and its 3-connected deletion minors are in $EX[E_4]$ and $EX[P_9^*]$.

$$F_7 = \left[\begin{array}{c|cccc} I_3 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 \end{array} \right] PG(3, 2) = \left[\begin{array}{c|cccccccccccc} I_4 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right]$$

In addition to these matroids, we prove that $EX[E_4]$ has the 17-element rank-5 internally 4-connected matroid R_{17} . This matroid appears in [3] as the extremal matroid for $EX[(K_5 \setminus e)^*]$. The class also has a 12-element rank-6 3-connected matroid M_{12} that is a splitter (every 3-connected single-element extension and coextension is not in the class). This matroid M_{12} is not internally 4-connected. The rest of the 3-connected matroids have a non-minimal exact 3-separation induced by P_9 or P_9^* . Matrix representations for M_{12} and R_{17} are shown below.

$$M_{12} = \left[\begin{array}{c|cccccc} I_6 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 1 & 1 \end{array} \right] R_{17} = \left[\begin{array}{c|cccccccccccccc} I_5 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{array} \right]$$

The next result is the main theorem in this paper.

Theorem 1.1. *Suppose M is a binary 3-connected non-regular matroid with no E_4 -minor. Then either P_9 or P_9^* are 3-decomposers for M or M or M^* is isomorphic to a 3-connected deletion-minor of Z_r for $r \geq 4$, F_7 , $PG(3, 2)$, R_{17} , or M_{12} .*

As an immediate corollary we can characterize $EX[P_9^*]$. The rank-5 internally 4-connected extremal matroid in $EX[P_9^*]$ is $R_{17} \setminus \{17\}$. For convenience of notation we call it R_{16} .

Corollary 1.2 *Suppose M is a binary 3-connected non-regular matroid with no P_9^* -minor. Then one of the following holds:*

- (i) M is isomorphic to F_7 , F_7^* , Z_r , Z_r^* , $Z_r \setminus b_r$, or $Z_r \setminus c_r$, for $r \geq 4$;
- (ii) P_9 is a 3-decomposer for M ; or
- (iii) M is isomorphic to a 3-connected deletion-minor of $PG(3, 2)$ or R_{16} .

By duality we get the following characterization for $EX[P_9]$.

Corollary 1.3 *Suppose M is a binary 3-connected non-regular matroid with no P_9 -minor. Then one of the following holds:*

- (i) M is isomorphic to F_7 , F_7^* , Z_r , Z_r^* , $Z_r \setminus b_r$, or $Z_r \setminus c_r$, for $r \geq 4$;
- (ii) P_9^* is a 3-decomposer for M ; or
- (iii) M is isomorphic to a 3-connected contraction-minor of R_{16} .

We end this section by describing our method for calculating extensions and coextensions. This method along with the Strong Splitter Theorem leads to relatively short proofs.

Let N be a $GF(q)$ -representable n -element rank- r matroid represented by the matrix $A = [I_r | D]$ over $GF(q)$. The columns of A may be viewed as a subset of the columns of the matrix that represents the projective geometry $PG(r-1, q)$. Let M be a simple single-element extension of N over $GF(q)$. Then $N = M \setminus e$ and M may be represented by $[I_r | D']$, where D' is the same as D , but with one additional column corresponding to the element e . The new column is distinct from the existing columns and has at least two non-zero elements. If the existing columns are labeled $\{1, \dots, r, \dots, n\}$, then the new column is labeled $(n+1)$.

Suppose M is a cosimple single-element coextension of N over $GF(q)$. Then $N = M/f$ and M may be represented by the matrix $[I_{r+1} | D'']$, where D'' is the same as D , but with one additional row. The new row is distinct from the existing rows and has at least two non-zero elements. The columns of $[I_{r+1} | D'']$ are labeled $\{1, \dots, r+1, r+2, \dots, n, n+1\}$. The coextension element f corresponds to column $r+1$. The coextension row is selected from $PG(n-r, q)$. We can visualize the new element f as appearing in the new dimension and lifting several points into the higher dimension. Observe that f forms a cocircuit with the elements corresponding to the non-zero elements in the new row. Note that in $[I_{r+1} | D'']$ the labels of columns beyond r are increased by 1 to accomodate the new column $r+1$.

We refer to the simple single-element extensions of N as Type (i) matroids and the cosimple single-element coextensions of N as Type (ii) matroids. The structure of Type (i) and Type (ii) matroids are shown in Figure 1.

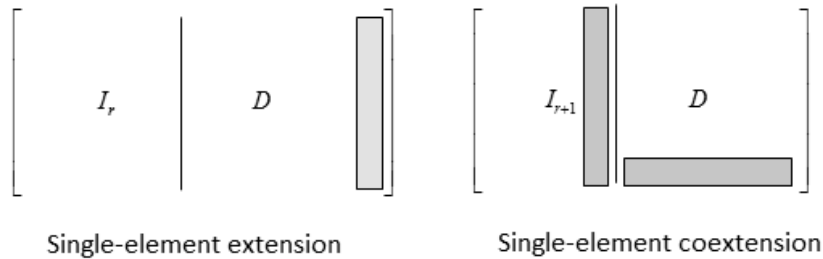
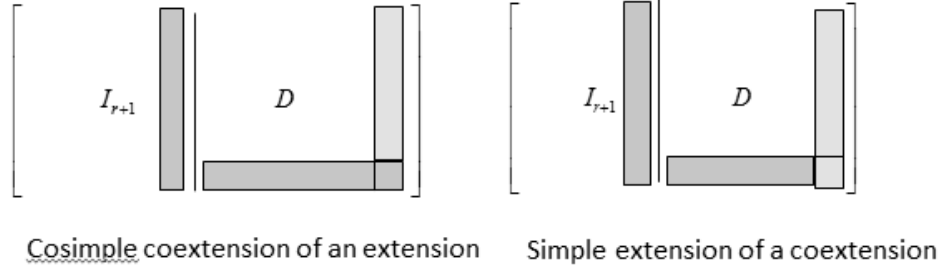


FIGURE 1. Structure of Type (i) and Type (ii) matroids

Once the simple single-element extensions (Type (i) matroids) and cosimple single-element coextensions (Type (ii) matroids) are determined, the number of permissible rows and columns give a bound on the choices for the cosimple single-element extensions of the Type (i) matroids and the simple single-element extensions of the Type (ii) matroids, respectively.

The structure of the cosimple single-element coextensions of a Type (i) matroid and the simple single-element extensions of a Type (ii) matroid are shown in Figure 2.

When computing the cosimple single-element coextension of a Type (i) matroid, there are three types of rows that may be inserted into the last row.

FIGURE 2. Structure of M , where $|E(M) - E(N)| = 2$

- (I) rows that can be added to N to obtain a coextension with a 0 or 1 as the last entry (or as many as the entries in $GF(q)$ for higher order fields);
- (II) the identity rows with a 1 in the last position; and
- (III) rows “in-series” to the right-hand side of the matrix with the last entry reversed.

When computing the simple single-element extension of a Type (ii) matroid, there are three types of rows that may be inserted into the last column.

- (I) columns that can be added to N to obtain an extension with a 0 or 1 as the last entry (or as many as the entries in $GF(q)$ for higher order fields);
- (II) the identity columns with a 1 in the last position; and
- (III) columns “in-parallel” to the right-hand side of matrix with the last entry reversed.

Suppose N' is a simple double-element extension of N formed by adding columns e_1 and e_2 and M is a cosimple single-element coextension of N' by element f . Then, by the Strong Splitter Theorem $M \setminus e_1$ or $M \setminus e_2$ is 3-connected except when $\{e_1, e_2, f\}$ is a triad. Thus, the only coextension of N' we must check is the one formed by adding row $[00 \dots 011]$. Additionally, there is no need to calculate the cosimple single-element extensions of the k -element extensions of N , for $k \geq 3$. This greatly reduces computations.

2. CHARACTERIZATION OF $EX[E_4]$

All proofs of excluded minor characterizations of binary non-regular matroids begin the same way. Tutte proved that a binary matroid is non-regular if and only if it has no minor isomorphic to F_7 or F_7^* [7, 10.1.2]. Observe that $F_7 = PG(2, 2)$ and as such has no extensions in the class of binary matroids. Coextensions of F_7 are duals of extensions of F_7^* . Thus we may focus on the extensions of F_7^* . Observe that $AG(3, 2)$ and S_8 are the two non-isomorphic 3-connected single-element extensions of F_7^* . Since they are self-dual, they are also the coextensions of F_7 . The matroid S_8 has two non-isomorphic 3-connected single-element extensions P_9 and Z_4 and $AG(3, 2)$ has one 3-connected single-element extension Z_4 . The simple extensions and cosimple-single-element coextensions of P_9 are given in Table 1a and 1b.

Matroid	Extension Columns	Name
P_9	[1110]	D_1
	[1001] [0101] [0110], [1010]	D_2
	[0011]	D_3
D_1	[0101] [0110] [1001] [1010]	X_1
	[0011]	X_2
D_2	[1010] [1110]	X_1
	[0011] [0101] [0110]	X_3
D_3	[1110]	X_2
	[0101] [0110] [1001] [1010]	X_3
X_1	[0011] [0101] [0110]	Y_1
	[1110]	Y_2
X_2	[0101] [0110] [1001] [1010]	Y_1
X_3	[0101] [0110] [1010] [1110]	Y_1

Table 1a: Rank 4 extensions of P_9

Coextension Rows	Name
[11000] [11111]	E_1
[11011] [11100]	E_2
[11001] [11101]	E_3
[01001] [01010] [01101] [01110] [10001] [10010] [10101] [10110]	E_4
[01011] [01100] [10011] [10100]	E_5
[00101] [00110]	E_6
[00111]	E_6^*
[00011]	E_7

Table 1b: Single-element coextensions of P_9

Suppose M is a 3-connected binary non-regular matroid with a P_9 -minor. From Tables 1a and 1b we see that P_9 has three non-isomorphic simple single-element extensions, D_1 , D_2 , and D_3 , and eight non-isomorphic cosimple single-element coextensions E_1 , E_2 , E_3 , E_4 , E_5 , E_6 , E_6^* , and E_7 .

$$\begin{aligned}
D_1 &= \left[\begin{array}{c|ccccc} I_4 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \right] & D_2 &= \left[\begin{array}{c|ccccc} I_4 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array} \right] & D_3 &= \left[\begin{array}{c|ccccc} I_4 & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array} \right] \\
E_1 &= \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right] & E_2 &= \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 1 & 1 \end{array} \right] & E_3 &= \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 1 & 1 \end{array} \right] & E_4 &= \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \\
E_5 &= \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 0 & 0 & 0 \end{array} \right] & E_6 &= \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right] & E_6^* &= \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right] & E_7 &= \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right]
\end{aligned}$$

Lemma 2.1. *If M has a P_9 - or P_9^* -minor, but no D_2 , D_2^* , E_4 , or E_5 -minor, then P_9 or P_9^* are 3-decomposers for M .*

Proof. The proof of this lemma appears in [3, Theorem 3.1] and is repeated here for convenience. Observe that P_9 has a non-minimal exact 3-separation (A, B) , where $A = \{1, 2, 5, 6\}$ is both a circuit and a cocircuit. It is easy to check that the set $A = \{1, 2, 5, 6\}$ is both a circuit and a cocircuit in D_1 and D_3 (note that every column is checked) whereas D_2 is internally 4-connected. The set $A = \{1, 2, 5, 6\}$ corresponds to $A' = \{1, 2, 6, 7\}$ in the coextension since the fifth column

is the coextended element. It can be checked that $\{1, 2, 6, 7\}$ is both a circuit and a cocircuit in $E_1, E_2, E_3, E_6, E_6^*$, and E_7 (every row is checked disregarding isomorphism). Further note that E_4 and E_5 are self-dual. Thus if M has a P_9 -minor, but no D_2, D_2^*, E_4 , or E_5 -minor, then P_9 or P_9^* is a 3-decomposer for M using Mayhew, Royle, and Whittle's sufficient one-element check [4]. \square

Proof of Theorem 1.1. Suppose M is a 3-connected binary matroid with no E_4 -minor. Lemma 2.1 implies that if M has a P_9 - or P_9^* -minor, but no D_2, D_2^*, E_4 , or E_5 -minor, then P_9 or P_9^* are 3-decomposers for M . So we must consider matroids that have an E_5, D_2 , or D_2^* -minor, but no E_4 -minor. Consider the 3-connected single-element extensions and coextensions of E_5 shown in Tables 2a and 2b.

Extension Columns	Name	E_4 -minor
[00101] [00110] [01011] [01100]	A	No
[10011]	B	No
[11001] [11101]	C	No
[00011] [00111] [01001] [01101]		Yes
01010] [01110]		Yes
[10001] [10010] [11011] [11100]		Yes
[10101] [10110] [11000] [11111]		Yes

Table 2a: Simple single-element extensions of E_5

Coextension Rows	Name
[00111] [01001] [01010] [01100]	A^*
[10011]	B^*
[10101] [11101]	C^*
[00011] [00101] [01011] [01101]	
[00110] [01110]	
[10001] [10010] [10111] [11100]	
[10100] [11001] [11010] [11111]	

Table 2b: Cosimple single-element coextensions of E_5

Observe that E_5 has seven non-isomorphic simple single-element extensions all of which have an E_4 -minor except A, B and C . Matrix representations for A, B , and C are given below.

$$A = \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad B = \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad C = \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

The next lemma is a key lemma for Theorem 1.1. It is also a useful stand-alone result for future structure theorems since E_4 and E_5 are important binary matroids. For this lemma we use the following representation of E_5 :

$$E_5 = \left[\begin{array}{c|ccccc} I_5 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \end{array} \right]$$

Lemma 2.2. *Suppose M is a binary 3-connected matroid with an E_5 -minor and no E_4 -minor. Then either $M \cong M_{12}$ or M or M^* is isomorphic to R_{17} or its 3-connected deletion-minors having an E_5 -minor.*

Proof. The proof is in three stages. First, we will show that all the coextensions of A , B , and C have an E_4 -minor with the exception of M_{12} . Suppose M is a coextension of A , B , C . The three types of rows that may be added to A , B and C to obtain M are:

- (I) rows that can be added to E_5 to obtain a coextension with no E_4 -minor, with a 0 or 1 as the last entry;
- (II) the identity rows with a 1 in the last position; and
- (III) the rows “in-series” to the right-hand side of matrices A , B , C with the last entry reversed.

Type I rows are $[001110]$, $[001111]$, $[010010]$, $[010011]$, $[010100]$, $[010101]$, $[011000]$, $[011001]$, $[100110]$, $[100111]$, $[101010]$, $[101011]$, $[111010]$, and $[111011]$. They are obtained from Table 2b. Type II rows are $[100001]$, $[010001]$, $[001001]$, $[000101]$, and $[000011]$. Type III rows are specific to the matrices A , B , C . For matrix A they are $[011111]$, $[101101]$, $[110110]$, $[111101]$, $[110000]$. For matrix B they are $[011110]$, $[101101]$, $[110111]$, $[111100]$, and $[110000]$. For C they are $[011110]$, $[101100]$, $[110111]$, $[111101]$, and 110000 .

Most of the above rows result in matroids that have an E_4 -minor (see bold-face rows in Table 3). Only a few coextensions must be specifically checked for an E_4 -minor. They are $(A, coextn11)$, $(B, coextn8)$, $(C, coextn8)$, $(C, coextn9)$, $(C, coextn10)$, $(C, coextn12)$, and $(C, coextn14)$.

Observe that $(A, coextn11)/11 \setminus 3 \cong E_4$, $(C, coextn8)/12 \setminus 2 \cong E_4$, $(C, coextn9)/12 \setminus 1 \cong E_4$, $(C, coextn10)/12 \setminus 10 \cong E_4$, and $(C, coextn14)/12 \setminus 6 \cong E_4$. Further, $(B, coextn8) \cong (C, coextn12)$ and this matroid does not have an E_4 -minor. This is the matroid M_{12} .

Second, we must establish that M_{12} is a splitter for $EX[E_4]$. By the Splitter Theorem and the fact that M_{12} is self-dual, we only need to check the single-element coextensions of M_{12} . From Table 3 observe that M_{12} as a coextension of C may be obtained by adding exactly one row. Thus, there are no further rows that may be added to form coextensions without an E_4 -minor. It follows that M_{12} is a splitter for the class of binary matroids with no E_4 -minor.

Third, we must show that either $M \cong M_{12}$ or $r(M) \leq 5$. To show this we compute the simple single-element extensions of A , B , and C with no E_4 -minor. From Table 2a the only columns that can be added to E_5 to obtain a matroid with no E_4 -minor are $[00101]$, $[00110]$, $[01011]$, $[01100]$, $[10011]$, $[11001]$, $[11101]$. They give the matroids D , E , F , and G shown below.

$$D = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ I_5 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \quad E = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ I_5 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

$$F = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ I_5 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array} \right] \quad G = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ I_5 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

Specifically, adding to A column $[00110]$, $[01100]$, or $[10011]$ gives D ; adding column $[01011]$ gives E ; adding $[11001]$ gives F ; and adding $[11101]$ gives G . Similarly, we can check that B extends to D and F and C extends to F and G . Observe that adding all seven columns to E_5 gives the 17-element matroid shown below which is isomorphic to the representation of R_{17} shown in the introduction. (Note that our strategy of throwing away all the matroids that can be decomposed

and directly obtaining the extremal internally 4-connected matroid side-steps having to calculate all the internally 4-connected matroids.)

$$R_{17} = \left[\begin{array}{c|cccccccccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{array} \right]$$

Matroid	Name	Coextension Row
<i>A</i>	coext 1	[000011] [000101] [001010] [011010] [101111] [111001]
	coext 2	[000110] [110011] [110101]
	coext 3	[000111] [101011] [111011]
	coext 4	[001001] [010110] [011111]
	coext 5	[001011] [011011] [100111]
	coext 6	[001100] [011100] [110000]
	coext 7	[001101] [010010] [010100] [011101] [101110] [111000]
	coext 8	[001110] [011000] [101101] [110010] [110100] [111101]
	coext 9	[001111] [011001] [100011] [100101] [101010] [111010]
	coext 10	[010001] [100010] [100100]
	coext 11	[010011] [010101] [100110]
	coext 12	[010111]
	coext 13	[100001] [101000] [111110]
	coext 14	[101001] [110110] [111111]
<i>B</i>	coext 1	[000011] [000101] [000110] [001001] [001010] [001111] [010010] [010100] [010111] [011000] [011011] [011110]
	coext 2	[000111] [001011] [010110] [011010]
	coext 3	[001100] [010001] [011101]
	coext 4	[001101] [001110] [010011] [010101] [011001] [011100]
	coext 5	[100001] [100010] [100100] [101000] [101101] [101110] [110000] [110011] [110101] [111001] [111100] [111111]
	coext 6	[100011] [100101] [101010] [101111] [111000] [111011]
	coext 7	[100110] [101001] [110010] [110100] [110111] [111110]
	coext 8	[100111] [101011] [111010]
<i>C</i>	coext 1	[000011] [000101] [001001] [001111] [010010] [010100] [011000] [011110] [100010] [100100] [101000] [101110] [110011] [110101] [111001] [111111]
	coext 2	[000110] [010111]
	coext 3	[000111] [010110] [100110] [110111]
	coext 4	[001010] [011011]
	coext 5	[001011] [011010] [101010] [111011]
	coext 6	[001100] [011101]
	coext 7	[001101] [011100] [101100] [111101]
	coext 8	[001110] [010011] [010101] [011001]
	coext 9	[010001]
	coext 10	[100001] [110000]
	coext 11	[100011] [100101] [101111] [111000]
	coext 12	[100111]
	coext 13	[101001] [110010] [110100] [111110]
	coext 14	[101011] [111010]

Table 3: Cosimple single-element coextensions of *A*, *B* and *C*

By the Strong Splitter Theorem, the only cosimple single-element coextensions of *D*, *E*, and *F* we must consider are the ones with [0000011] as the new row. Let us call them *D'*, *E'*, *F'*, and *G'*, respectively. In each case we can find an E_4 minor. In particular, $D'/1 \setminus \{3, 11\} \cong E_4$, $E'/1 \setminus \{7, 11\} \cong E_4$, $F'/1 \setminus \{3, 11\} \cong E_4$, and $G'/1 \setminus \{7, 11\} \cong E_4$. \square

Returning to the proof of Theorem 1.1, it remains to show that if *M* has a D_2 -minor and no E_4 -minor, then we do not get any new matroids other than those already found in Lemma 2.2.

Suppose *M* is a cosimple single-element coextension of D_2 . From Table 4 we see that *M* is isomorphic to *A*, *B*, *C*, or *Z*. A matrix representation for *Z* is shown below:

$$Z = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

Since Z is formed by adding only one row to D_2 (namely $[000111]$) any coextension of Z will also be a coextension of A , B , and C .

Suppose M is a single-element extension of D_2 . From Table 1a we see that that D_2 has two single-element extensions X_1 and X_3 shown below:

$$X_1 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ I_4 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{array} \right] \quad X_3 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ I_4 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{array} \right]$$

By the Strong Splitter Theorem the only coextensions of X_1 and X_2 we must check are the ones formed with $[0000011]$ as the new row. Both these matroids have an E_4 -minor.

Lastly, suppose M is a simple single-element extension of Z . It is straightforward to compute the three non-isomorphic simple single-element extensions which are D , F and Y (obtained by adding one of columns $[00111]$, $[01011]$, $[01101]$, $[10101]$, or $[11100]$). The result follows again by Lemma 2.2 and the fact that when we add the above six columns to Z we get the sixteen element matroid shown below which is isomorphic to $R_{17} \setminus \{17\}$.

$$R_{16} = \left[\begin{array}{c|cccccccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array} \right]$$

Thus Z does not contribute any new matroids to $EX[E_4]$ other than those found in Lemma 2.2. This completes the proof of Theorem 1.1. \square

Matroid	Coextension Rows	Name	Relevant minors
D_2	$[000011]$ $[000101]$ $[000110]$ $[001111]$ $[100111]$ $[101000]$	A_{26} A	E_5, E_6^*, E_7
	$[000111]$	A_{31} Z	E_7, R_{10}
	$[001001]$ $[010100]$ $[011101]$	A_{23}	E_4, E_5
	$[001010]$ $[001100]$ $[010001]$ $[010010]$ $[011011]$ $[011110]$	A_{20}	E_4, E_6
	$[001011]$ $[001101]$ $[010101]$ $[010110]$ $[011001]$ $[011100]$	A_{21}	E_4, E_5
	$[001110]$ $[010011]$ $[011010]$	A_{24}	E_4
	$[100001]$ $[101000]$ $[101011]$ $[101101]$ $[110110]$ $[111001]$	A_{15}	E_2, E_5
	$[100010]$ $[100100]$ $[110000]$ $[110101]$ $[111100]$ $[111111]$	A_6	E_1, E_4
	$[100011]$ $[100101]$ $[110010]$ $[110111]$ $[111000]$ $[111011]$	A_{16}	E_2, E_3, E_4, E_6^*
	$[100110]$ $[101010]$ $[101100]$ $[101111]$ $[110001]$ $[111110]$	A_7	E_4, E_5
	$[100111]$ $[110011]$ $[111010]$	A_{18} C	E_3, E_5, E_6^*, E_7
	$[101001]$	A_{27} B	E_5

Table 4: Cosimple single-element coextensions of D_2

Proof of Corollary 1.2. Suppose M is a 3-connected binary non-regular matroid with a P_9 -minor, but no P_9^* -minor. Lemma 2.1 implies that P_9 is a 3-decomposer or M has a minor isomorphic to D_2 or D_2^* since E_4 and E_5 have a P_9^* -minor. Observe that D_2 has four non-isomorphic cosimple single-element coextensions A , B , C , and Z . Of these A , B , and C have an E_5 -minor,

and therefore a P_9^* -minor. Thus, if M is a cosimple single-element coextension of D_2 , then $M \cong Z$. Suppose M is a simple single-element extension of Z , then $M \cong R_{16}$ as explained in the proof of Theorem 1.1. \square

By duality we also have a decomposition result for $EX[P_9]$. Note that for $EX[P_9]$ among the rank 4 matroids only F_7^* , S_8 , $AG(3, 2)$ or Z_4 have no P_9 -minor. All the others have a P_9 -minor. Moreover, $S_8 \cong Z_4 \setminus b_r$ and $AG(3, 2) \cong Z_4 \setminus c_r$, so we do not have to explicitly list them in the statement of the corollary.

Once the 3-decomposers and internally 4-connected extremal matroids are known for an excluded minor class, it is easy to determine all the internally 4-connected members of the class. Observe that F_7 and F_7^* are internally 4-connected by default. All the restrictions of $PG(3, 2)$ of size 10 and higher are internally 4-connected with the exception of D_1 , D_3 and X_2 (8 matroids in total). The restrictions of R_{16} in $EX[P_9^*]$ that are internally 4-connected are $R_{16} \setminus \{16\}$, $R_{16} \setminus \{15, 16\}$, $R_{16} \setminus \{14, 15, 16\}$, $R_{16} \setminus \{13, 14, 15, 16\}$, $R_{16} \setminus \{12, 13, 14, 15, 16\}$, and R_{10} (six matroids in total). Thus there are 16 binary internally 4-connected matroids in $EX[P_9^*]$ and 8 in $EX[P_9]$.

We end this paper with a brief description of the usefulness of the Strong Splitter Theorem. The reader will note that it was used twice in the proof of Theorem 1.1. As a consequence we had to do calculations up to only rank 6 and 13 elements and that too for very few matroids. Moreover, if needed, the reader can easily verify all the calculations. This is because once a required minor is found, checking that the specified deletions and contractions do indeed give the minor is very easy. Similarly for isomorphism. Once the isomorphism is found, it is easy to verify. The proof in this paper is not a “computer-proof.”

In [4] and [5] the authors used Chun, Mayhew, and Oxley’s chain theorem for internally 4-connected binary matroids [1]. According to this chain theorem if a counterexample exists, then it has to be at most three more elements than one of the known internally 4-connected matroids in the class. So, for example, in [5] the authors extend and coextend known internally 4-connected matroids up to 3 more elements to see if any new internally 4-connected matroids appear. They had to do an exhaustive search up to rank 8 and 20 elements. Their proof is longer and had a component that relied solely on a computer check.

Had we not used the Strong Splitter Theorem and instead used Chun, Mayhew, and Oxley’s result from [1], we also would have had to do an exhaustive search up to rank 8 and 20 elements. What’s worse we wouldn’t have found the splitter M_{12} because it is not internally 4-connected, yet it has no 3-separation induced by the one in P_9 or P_9^* . That makes it an important member of the class. Due to the efficient stair-stepping approach of the Strong Splitter Theorem, we had to coextend only four rank-5 matroids by 1 element (so the rank is at most 6) and extend by 2 elements up to 13 elements. Moreover, only four rank-6, 13-element matroids had to be checked for the required minor.

Using the Strong Splitter Theorem gives an enormous savings in terms of computation, explains why an excluded minor class like $EX[E_4]$ lends itself to a concise characterization, and most importantly gives a short proof of a decomposition result which is stronger than a result that identifies all the internally 4-connected matroids. A complete list of all the internally 4-connected matroids follows immediately from a decomposition theorem like Theorem 1.1.

References

- (1) Chun, C., Mayhew, D. and Oxley, J. (2011) A chain theorem for internally 4-connected binary matroids. *J. Combin. Theory Ser. B* 101, no. 3, 141 - 189.

- (2) Kingan, S. R. and Lemos, M. (2014). Strong Splitter Theorem, *Annals of Combinatorics* **18-1**, 111-116.
- (3) Kingan, S. R. and Lemos, M. (to appear). Decomposition of binary matroids with no prism minor, *Graphs and Combinatorics*.
- (4) Mayhew, D., Royle, G. and Whittle, G (2011). *The internally 4-connected binary matroids with no $M(K_{3,3})$ -minor*, Memoirs of the American Mathematical Society, **981**, American Mathematical Society, Providence, Rhode Island.
- (5) D. Mayhew and G. Royle (2012). The internally 4-connected binary matroids with no $M(K_5 \setminus e)$ -minor, *Siam Journal on Discrete Mathematics*, **26**, 755-767.
- (6) Oxley, J. G. (1987) The binary matroids with no 4-wheel minor, *Trans. Amer. Math. Soc.* **154**, 63-75.
- (7) Oxley, J. G. (2012) *Matroid Theory*, Second Edition, Oxford University Press, New York.
- (8) Seymour, P. D. (1980) Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28**, (1980) 305-359.